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Engineering Notes

Minimum-Cost Cruise at Constant Altitude of Commercial Aircraft Including Wind Effects

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I. Introduction

AIRCRAFT trajectory optimization is a subject of great importance in air traffic management from the point of view of defining optimal flight procedures that lead to energy-efficient flights. In practice, the airlines consider a cost index (CI) and define the direct operating cost (DOC) as the combined cost of fuel consumed and flight time weighted by the CI. Their goal is to minimize the DOC. However, in the presence of unexpected winds, the flight time may differ considerably from the scheduled time, which leads to an arrival-error cost that can be added to the DOC to obtain the total cost (TC).

Minimum-DOC trajectories have been studied by different authors [1–6]. The related problem of minimum fuel with fixed final time has been analyzed as a minimum-DOC problem with free final time in [3,5,7] (the problem is to find the time cost for which the corresponding free final time DOC-optimal trajectory arrives at the assigned time); this same problem is addressed in [8], analyzing the effects of mismodeled winds in a scenario formed by the final cruise and descent segments. The problem of minimum-cost flight, considering not only the DOC but also the arrival-error cost, is analyzed in [9,10], taking into account factors such as crew overtime cost, passenger dissatisfaction cost, and losses due to missed connections.

In this Note, the problem of minimum-cost cruise at constant altitude in the presence of strong winds, including the arrival-error cost, is analyzed, considering the general unsteady problem, with variable aircraft mass, and without any restriction on cruise altitude. The main objective is to analyze the optimal trajectories that lead to minimum cost, defined as optimal speed laws (speed as a function of aircraft mass). The analysis is made using the theory of singular optimal control (see [11]), which has the great advantage of providing feedback control laws (control variables as functions of the state variables) that can be directly used to guide the aircraft along the optimal path. These optimal control laws are analyzed as well.

In this work, the initial and final speeds are given, so that the optimal control is of the bang-singular-bang type, and the optimal paths are formed by a singular arc and two minimum/maximum-thrust arcs joining the singular arc with the given initial and final points (see [6,12]). In previous work related to optimum cruise at constant altitude [13,14], only the singular arc was studied; hence, a more general formulation of the optimal problem is addressed now, apart from considering the arrival-error cost and including wind effects (average horizontal winds).

In this analysis of the minimum-TC problem, the arrival-error cost depends on the difference between the actual and the scheduled flight times, and it is defined to be positive, so that both late and early arrivals are penalized (the objective is to achieve high arrival-time accuracy). It will be shown that, for some values of the parameters of the problem, minimum cost is obtained when the final time coincides with the scheduled time of arrival; that is, when the arrival-error cost is zero. This critical case is in fact a problem with fixed final time. Results are presented for a model of a Boeing 767-300ER.

II. Problem Formulation

In this section, the optimal control problem is formulated. The DOC can be defined as $m_F + \mathrm{CI}t_f$, where m_F is the mass of fuel consumed and t_f is the flight time. Note that, for simplicity, the actual fuel cost has not been included, so that the DOC is scaled to the units of fuel mass (using SI units of measure, the DOC is measured in kilograms and the CI in kilograms per second; representative values of CI are in the range of 0.5 to 1.5 kg/s).

The arrival-error cost is defined as $\Phi(t_f) = K|t_f - t_s|$, where t_s is the scheduled time of arrival and K is a positive constant parameter that must be defined by the airline (K is measured in kilograms per second). The reference flight condition considered in this Note to define t_s corresponds to minimum DOC, no wind (NW), and the given value of CI. This arrival-error cost (which is defined to be positive) penalizes both late and early arrivals, so that one can accomplish the objective of achieving high arrival-time accuracy in the case of strong winds. Note that Φ can be also written as

$$\Phi(t_f) = \Phi_{t_f}(t_f - t_s) \tag{1}$$

with

$$\Phi_{t_f} = \begin{cases} +K & \text{if } t_f > t_s \\ -K & \text{if } t_f < t_s \end{cases} \tag{2}$$

The objective is to minimize the TC; that is, to minimize the following performance index:

$$\mathrm{TC} = \int_0^{t_f} (cT + \mathrm{CI})\,\mathrm{d}t + \Phi_{t_f}(t_f - t_s) \tag{3}$$

(where the integral corresponds to the DOC), subject to the following constraints:

$$\dot{V} = \frac{1}{m}(T - D) \qquad \dot{m} = -cT \qquad \dot{x} = V + w \tag{4}$$

which are the equations of motion for cruise at constant altitude and constant heading, with a constant horizontal wind. In these equations, the drag is a general known function $D(V, m)$, which takes into account the remaining equation of motion $L = mg$; the thrust $T(V)$ is given by $T = \pi T_M(V)$, where π models the throttle setting, $0 < \pi_{\min} \leq \pi \leq \pi_{\max} = 1$, and $T_M(V)$ is a known function. The specific fuel consumption $c(V)$ is also a known function, and the windspeed w is a known parameter. Thus, in this problem, there are three states (speed V, aircraft mass m, and distance x) and one control (π). The initial values of speed, aircraft mass, and distance (V_i, m_i, and x_i), and the final values of speed and distance (V_f and x_f) are given. The final value of aircraft mass m_f and the flight time t_f are unspecified. (Note that D, T_M, and c also depend on the given altitude h.)

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A. Necessary Conditions for Optimality

The Hamiltonian of this problem is given by

$$H = (c\pi T_M + \mathrm{CI}) + \frac{\lambda_V}{m}(\pi T_M - D) - \lambda_m c\pi T_M + \lambda_x(V + w) \quad (5)$$

where λ_V, λ_m, and λ_x are the adjoint variables, which are defined by the following equations:

$$\dot{\lambda}_V = -\frac{\partial H}{\partial V} = -\lambda_x + \frac{\lambda_V}{m}\frac{\partial D}{\partial V} - \left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right]\pi\frac{\mathrm{d}T_M}{\mathrm{d}V} + (\lambda_m - 1)\frac{\mathrm{d}c}{\mathrm{d}V}\pi T_M$$

$$\dot{\lambda}_m = -\frac{\partial H}{\partial m} = \frac{\lambda_V}{m}\left[\frac{\pi T_M - D}{m} + \frac{\partial D}{\partial m}\right] \qquad \dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \quad (6)$$

The last equation leads to $\lambda_x = \text{constant}$. These equations are necessary conditions for optimality. One also has the following transversality conditions (which are necessary conditions for optimality as well): first, since the final value of the aircraft mass $m(t_f)$ is not specified, one has

$$\lambda_m(t_f) = 0 \quad (7)$$

and second, since the final time is not specified, one has

$$H(t_f) = -\frac{\partial \Phi}{\partial t_f} = -\Phi_{t_f} \quad (8)$$

It still remains to formulate the Hamiltonian minimization condition (see [15]) that states that, for the control to be optimal, it is necessary that it globally minimize the Hamiltonian. The Hamiltonian is linear in the control variable, so that one can write $H = \bar{H} + S\pi$, with

$$\bar{H} = \mathrm{CI} - \lambda_V\frac{D}{m} + \lambda_x(V + w) \qquad S = \left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right]T_M \quad (9)$$

where S is the switching function. Thus, since H is linear in π, and π is bounded, minimization of H with respect to π defines the optimal control as follows:

$$\pi = \begin{cases} \pi_{\max} & \text{if } S < 0 \\ \pi_{\min} & \text{if } S > 0 \\ \pi_{\text{sing}} & \text{if } S = 0 \text{ over a finite time interval} \end{cases} \quad (10)$$

where π_{sing} ($\pi_{\min} < \pi_{\text{sing}} < \pi_{\max}$) is the singular control (yet to be determined). A necessary condition for the singular control to be optimal is analyzed next in Sec. III.A.

Moreover, since the Hamiltonian is not an explicit function of time, one has the first integral $H = \text{constant}$ on the optimal trajectory, and using Eq. (8), one gets

$$H(t) = -\Phi_{t_f} \quad (11)$$

That is, the Hamiltonian is constant along the optimal trajectory ($-K$ for late arrivals and $+K$ for early arrivals).

B. Optimal Trajectories

In general, the optimal trajectories will be composed of singular arcs (arcs with π_{sing}) and arcs with $\pi_{\min}$ or $\pi_{\max}$; whether one has $\pi_{\min}$ or $\pi_{\max}$ is defined by the sign of the switching function S. In this problem, one expects the solution to be of the bang-singular-bang type. (See [6], in which, using a reduced-order model with different time scales, the trajectory is decomposed into three parts: an initial transient, the cruise-dash arc, and a terminal transient. A different approach is followed in [16], in which the trajectory structure for the Goddard's problem is obtained by means of continuation techniques). Since the initial and final speeds are fixed, there is a physical criterium to decide whether one has $\pi_{\min}$ or $\pi_{\max}$ by just comparing those speeds with the speeds that correspond to the singular arc. (Although called optimal trajectories, they are in fact extremals; that is, they are trajectories that satisfy the necessary conditions for optimality.)

A necessary condition for the junctions between singular and nonsingular arcs to be optimal is analyzed next in Sec. III.B.

III. Singular Arc

The singular control is obtained when the switching function is zero ($S = 0$) on an interval of time; hence, since $H = -\Phi_{t_f}$, one also has $\bar{H} = -\Phi_{t_f}$. On that interval of time, one has $\dot{S} = 0$ as well. The singular arc is defined by the three equations (see [12]):

$$\bar{H} + \Phi_{t_f} = S = \dot{S} = 0 \quad (12)$$

The function $\dot{S}$ is given by

$$\dot{S} = -\left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right]\frac{D}{m}\frac{\mathrm{d}T_M}{\mathrm{d}V} + \left[\frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V} + cD - mc\frac{\partial D}{\partial m}\right) - \lambda_x + (\lambda_m - 1)D\frac{\mathrm{d}c}{\mathrm{d}V}\right]\frac{T_M}{m} \quad (13)$$

(Note that the terms in the control variable π have canceled out of this equation.)

Hence, the three equations that define the singular arc (12) reduce to

$$\mathrm{CI} + \Phi_{t_f} - \lambda_V\frac{D}{m} + \lambda_x(V + w) = 0 \qquad \frac{\lambda_V}{m} - (\lambda_m - 1)c = 0$$

$$\frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V} + cD - mc\frac{\partial D}{\partial m}\right) - \lambda_x + (\lambda_m - 1)D\frac{\mathrm{d}c}{\mathrm{d}V} = 0 \quad (14)$$

The singular arc is obtained after eliminating the adjoints, λ_V and λ_m, from these equations. One obtains the following expression:

$$D\left[\left(1 - \frac{\Omega}{\Omega + V}\right) - Vc - \frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right] - V\frac{\partial D}{\partial V} + Vcm\frac{\partial D}{\partial m} = 0 \quad (15)$$

which is a family of singular arcs of parameter $\Omega = [(\mathrm{CI} + \Phi_{t_f})/\lambda_x] + w$, namely, $f(V, m, \Omega) = 0$. The unknown constant λ_x is defined by the condition $\lambda_m(t_f) = 0$; hence, to determine λ_x, the adjoint equations must be integrated along with the state equations (the numerical procedure is described in Sec. IV). Once λ_x is determined, Eq. (15) defines a singular line in the (V, m) space, which is in fact the locus of possible points in the state space where singular arcs must lie. This singular line is also a switching boundary for the optimal control (see [17]).

Note that, for the particular case of minimum-fuel cruise ($\mathrm{CI} = 0$), with $w = 0$ and $K = 0$, one has the same singular arc as the one obtained in [13] for the problem of maximum-range cruise. Also note that this family of singular arcs is the same as that obtained in [14] for the problem of minimum-fuel cruise with given final time (but with a different family parameter); therefore, for given t_f, the minimum-fuel problem is equivalent to solving a minimum-DOC problem with the appropriate value of the family parameter, as indicated in [3,5,7].

A. Optimal Singular Control

The function $\ddot{S}$ depends linearly on the control variable π. Since this second total derivative of S is the lowest-order derivative in which π appears explicitly, the order of the singular arc is $q = 1$ (in general, the order is q when such derivative is of order $2q$, as defined in [11]).

Let $\ddot{S} = A(V, m) + B(V, m)\pi$; therefore, because one also has $\ddot{S} = 0$ (where $S = \dot{S} = 0$), the singular control is obtained from $A(V, m) + B(V, m)\pi = 0$, and one gets the following:

$$\pi_{\text{sing}} = \frac{D}{T_M}\left(1 + Vc\frac{A_1(V, m)}{B_1(V, m)}\right) \tag{16}$$

where $A_1(V, m)$ and $B_1(V, m)$ are given by

$$\begin{aligned} A_1(V, m) &= m\frac{\partial^2 D}{\partial m \partial V} - m^2 c \frac{\partial^2 D}{\partial m^2} - mc\frac{\partial D}{\partial m} \\ &\quad - \frac{m}{D}\frac{\partial D}{\partial m}\left(\frac{\partial D}{\partial V} - mc\frac{\partial D}{\partial m}\right) \\ B_1(V, m) &= DV\left(c^2 + 3\frac{\mathrm{d}c}{\mathrm{d}V} + \frac{1}{c}\frac{\mathrm{d}^2 c}{\mathrm{d}V^2}\right) + 2\frac{\partial D}{\partial V}\left(Vc + \frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right) \\ &\quad - mV\left(c^2 + 3\frac{\mathrm{d}c}{\mathrm{d}V}\right)\frac{\partial D}{\partial m} + V\frac{\partial^2 D}{\partial V^2} + m^2 c^2 V \frac{\partial^2 D}{\partial m^2} - 2Vcm\frac{\partial^2 D}{\partial m \partial V} \end{aligned} \tag{17}$$

This expression for the optimal singular control depends implicitly on the parameter of the family of singular arcs, since V and m are related by the singular arc equation (15), which includes the dependence on Ω.

The generalized Legendre–Clebsch condition (see [18]) establishes that, for the singular control to be optimal, one must have

$$(-1)^q \frac{\partial}{\partial \pi}\left[\frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}} S\right] \geq 0 \tag{18}$$

which, in our case ($q = 1$), reduces to

$$-\frac{\partial \ddot{S}}{\partial \pi} \geq 0 \tag{19}$$

which can be shown to be satisfied numerically for the aircraft model considered in this Note; in fact, it can be shown that the strengthened generalized Legendre–Clebsch condition $(-(\partial \ddot{S}/\partial \pi) > 0)$ is satisfied.

B. Optimality of Junctions

For the optimality of junctions between singular and nonsingular arcs, the following necessary condition must be satisfied (see [19]): the sum of the order of the singular arc and the lowest-order time derivative of the control that is discontinuous at the junction must be an odd integer if the strengthened generalized Legendre–Clebsch condition is satisfied at the junction and if the control is piecewise analytic in a neighborhood of the junction (which is the case in this work). This necessary condition is shown to be satisfied, since the order of the singular arc is $q = 1$ and the lowest-order time derivative of the control that is discontinuous at the junction is $r = 0$ (that is, the control itself is discontinuous at the junction).

Moreover, note that, at the junctions, where the control variable is discontinuous, the adjoint variables, the Hamiltonian, and the switching function are all continuous; hence, the Weierstrass–Erdman corner conditions are satisfied (see [12]).

IV. Numerical Procedure

In Fig. 1, a sketch of the expected optimal path (bang-singular-bang) is presented. Knowing the structure of the solution allows one to define an efficient numerical procedure (see [20]). The first bang starts with the following initial values: $V(0) = V_i$, $m(0) = m_i$, and $x(0) = 0$. The state equations can be integrated until the singular arc

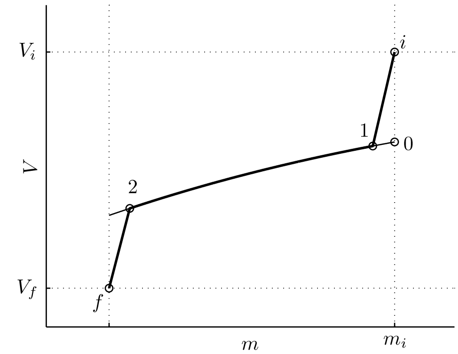


Fig. 1 Sketch of optimal path.

$f(V, m, \Omega) = 0$ is reached (note that Ω is unknown because λ_x is unknown). If λ_x were known, the state equations could be integrated along the singular arc from point 1 to point 2 (see Fig. 1), but the corresponding distance x_{12} is unknown. If x_{12} were known, the state equations could be integrated along the second bang, which starts at the singular arc (point 2) and ends when the value $V = V_f$ is reached. At that final point, one has two additional conditions, $x(t_f) = x_f$ and $\lambda_m(t_f) = 0$, which must be used to define λ_x and x_{12} (namely, the two unknown parameters in the resolution process); this task is performed by means of an iterative procedure.

This iterative procedure must be started with an initial guess. From Eqs. (14), one gets the expression

$$\lambda_x = -\frac{\mathrm{CI} + \Phi_{t_f} + (1 - \lambda_m)cD}{V + w} \tag{20}$$

which, particularized at point 2 with $\lambda_{m_2} \approx 0$ (recall that $\lambda_m = 0$ at the final point, which is very close to point 2) leads to

$$\lambda_x^{[0]} = -\frac{\mathrm{CI} + \Phi_{t_f} + c_2 D_2}{V_2 + w} \tag{21}$$

For V_2, c_2 and D_2 representative values are taken, obtained with $M_2 = 0.75$ and $m_2 = 115{,}000$ kg (for our particular aircraft model). The initial guess for x_{12} is $x_{12}^{[0]} = x_f$, since the two bang arcs have very small lengths.

To have a positive arrival-error cost, for headwinds (HWs), one selects $\Phi_{t_f} = +K$ (because one expects $t_f > t_s$) and, for tailwinds (TWs), $\Phi_{t_f} = -K$ (because one expects $t_f < t_s$). Once t_f has been calculated, one must check that the condition $\Phi_{t_f}(t_f - t_s) > 0$ is satisfied. If not, one has the type of critical problem studied next in Sec. V.

It still remains to check whether the assumed structure for the control (bang-singular-bang) is correct. That is, one must check that $S > 0$ for $\pi = \pi_{\min}$ and $S < 0$ for $\pi = \pi_{\max}$. This requires the computation of S along the extremal path; to do that, λ_V and λ_m are computed by backward integration along the first bang (from point 1, with known values λ_{V_1} and λ_{m_1}), and by forward integration along the final bang (from point 2, with known values λ_{V_2} and λ_{m_2}). The numerical results show that the control structure is correct in all cases.

V. Critical Minimum-Cost Problem

It will be shown next that, for some values of the parameters, the numerical procedure just described yields results that satisfy $\Phi_{t_f}(t_f - t_s) < 0$ (with $\Phi_{t_f} < 0$, one has $t_f > t_s$, and if one changes to $\Phi_{t_f} > 0$, one gets $t_f < t_s$), which contradicts the assumption of positive arrival-error cost. Since one cannot have $t_f > t_s$ nor $t_f < t_s$, the only possibility is to have $t_f = t_s$; that is, optimum performance is obtained by making the arrival-error cost zero. Therefore, the problem becomes one with fixed final time. This same result has been reported by Chakravarty [9] in the analysis of global trajectories (climb-cruise-descent).

To analyze this behavior further, the free final time minimum-cost problem is transformed in a series of fixed final time problems, so that the solution is defined by the particular value of t_f that yields the minimum value of the fixed final time minimum-cost function (see [12]). Hence, the minimum-cost problem is formulated in the following equivalent form:

$$\mathrm{TC}_{\min} = \min_{t_f} J(t_f) \tag{22}$$

where

$$J(t_f) = \min_{\pi} [m_F + \mathrm{CI} t_f + \Phi(t_f)] \tag{23}$$

with the minimization constrained by Eq. (4), and with t_f fixed.

The following auxiliary problem is also considered:

$$\mathrm{DOC}_{\min} = \min_{t_f} J'(t_f) \tag{24}$$

where

$$J'(t_f) = \min_{\pi} (m_F + \mathrm{CI} t_f) \tag{25}$$

with the minimization constrained by Eq. (4), and with t_f fixed.

Note that $J(t_f) = J'(t_f) + \Phi(t_f)$; thus, if $J'(t_f)$ has a relative minimum at $t_f < t_s$, then $J(t_f)$ has a minimum at $t_f \le t_s$ (it cannot be at $t_f > t_s$, because both $J'(t_f)$ and $\Phi(t_f)$ are increasing functions). Analogously, if $J'(t_f)$ has a relative minimum at $t_f > t_s$, then $J(t_f)$ has a minimum at $t_f \ge t_s$ (it cannot be at $t_f < t_s$, because both $J'(t_f)$ and $\Phi(t_f)$ are decreasing functions). Note that $J(t_f)$ is not derivable at $t_f = t_s$ (it is a critical point).

The critical case corresponds to the problem in which $J(t_f)$ does not have a relative minimum; in such a case, the minimum must be at the critical point (the end of the interval); that is, it must be at $t_f = t_s$. Thus, one has that the optimal solution corresponds to a problem of minimum cost (in fact, minimum fuel consumption) with fixed arrival time.

To illustrate this analysis, Fig. 2 displays the results for two cases: one in which there is a relative minimum (for $\mathrm{CI} = 1$ kg/s, $h = 10{,}000$ m, $K = 0.5$ kg/s, and TW $w = 20$ m/s), and another one in which one has the critical case (for $\mathrm{CI} = 0$, $h = 9000$ m, $K = 0.5$ kg/s, and TW $w = 20$ m/s). The solid lines in the figure correspond to $J(t_f)$, and the dashed lines correspond to $J'(t_f)$. In these graphs, by comparing the results for minimum DOC (points A) and for minimum TC (points B), one can see that the optimal value of t_f increases when the arrival-error cost Φ is taken into account and that the minimum cost increases (the solution moves from point A to point B).

VI. Results

The aircraft model considered in this Note for the numerical applications is that of a Boeing 767-300ER (a typical twin-engine widebody long-range transport aircraft), which is described in [14]. With respect to the atmosphere, the International Standard Atmosphere model is considered.

Results are presented for a cruise flight defined by the range $x_f = 10{,}000$ km, which is representative of an extended-range aircraft, and by initial and final speeds $V_i = 240$ m/s and $V_f = 180$ m/s (the same values in all cases studied next). Except when the effect of windspeed is analyzed, results are presented for a HW $w = -20$ m/s and a TW $w = 20$ m/s. Even though, in the HW case, one would normally have more initial fuel than in the TW case, for simplicity, the initial aircraft weight at the start of the cruise is the same in all cases: $W_i = 1600$ kN.

A. Minimum-TC Problem

In this section, the problem of minimum TC is analyzed for HWs and TWs (including, for comparison, results for the case of NW). As indicated in Sec. IV, for the case of HWs, one selects $\Phi_{t_f} = +K$, and for the case of TWs, one selects $\Phi_{t_f} = -K$, so that one has a positive arrival-error cost $[\Phi_{t_f}(t_f - t_s) > 0]$. In the following, results for $K = 0.5$ kg/s are presented, except when the influence of K on the results is analyzed. (The problem with NW corresponds to $K = 0$ by definition.)

1. Optimal Trajectories and Optimal Control

The optimal trajectories, in the W-M plane (Mach number vs aircraft weight) are represented in Fig. 3 for $h = 10{,}000$ m, for two values of CI (0 and 1 kg/s), and for the wind conditions given previously. These trajectories are composed of three arcs; first, there is a minimum or maximum-thrust arc, followed by a singular arc and, finally, a minimum-thrust arc (for the case of $\mathrm{CI} = 1$ kg/s and HW, the values of the Mach numbers for the singular arc are greater than the initial value; therefore, the optimal trajectory starts with a maximum-thrust arc). Note that, for some combinations of CI and wind condition, the singular arc defines a flight segment of almost constant Mach number.

As expected, for a given CI, the optimal values of Mach numbers are larger for the case of HW and smaller for TW. The differences between HWs and TWs are larger for lower values of the CI. Also

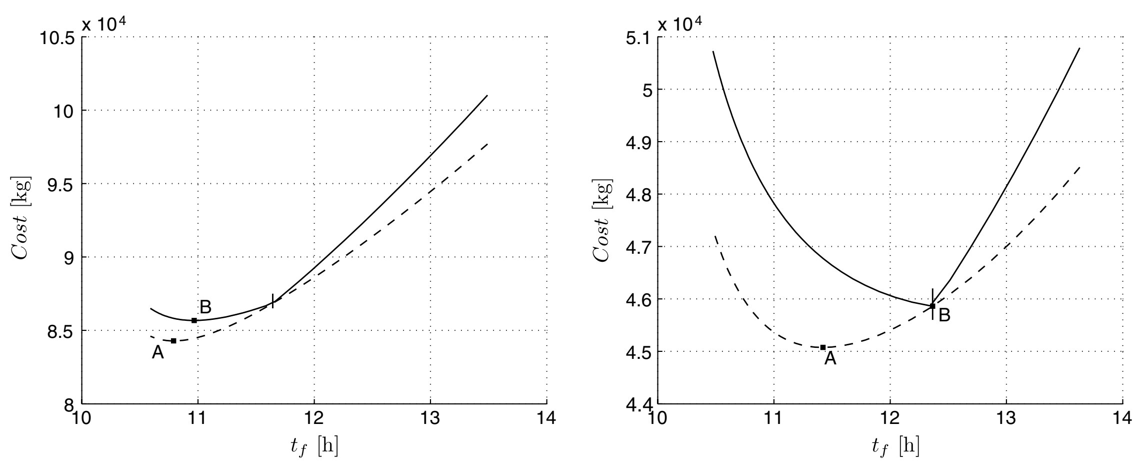
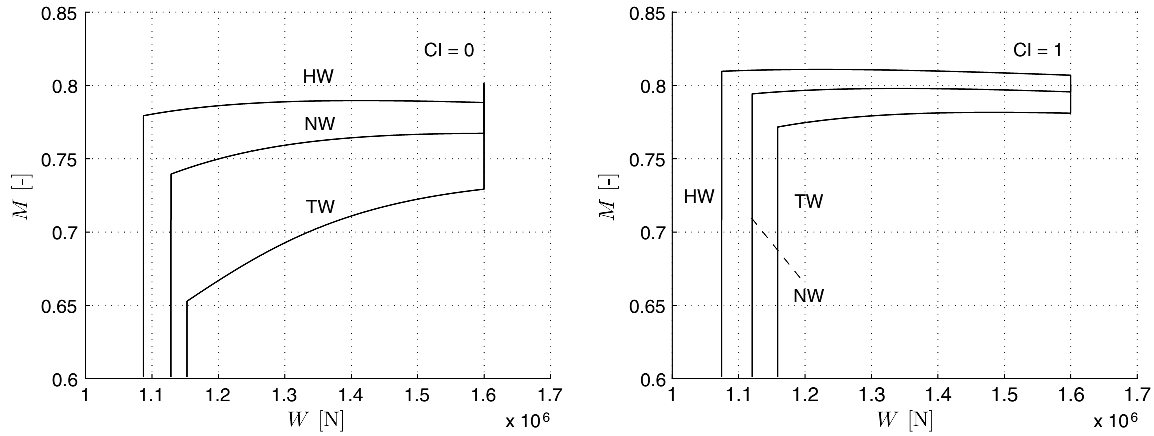
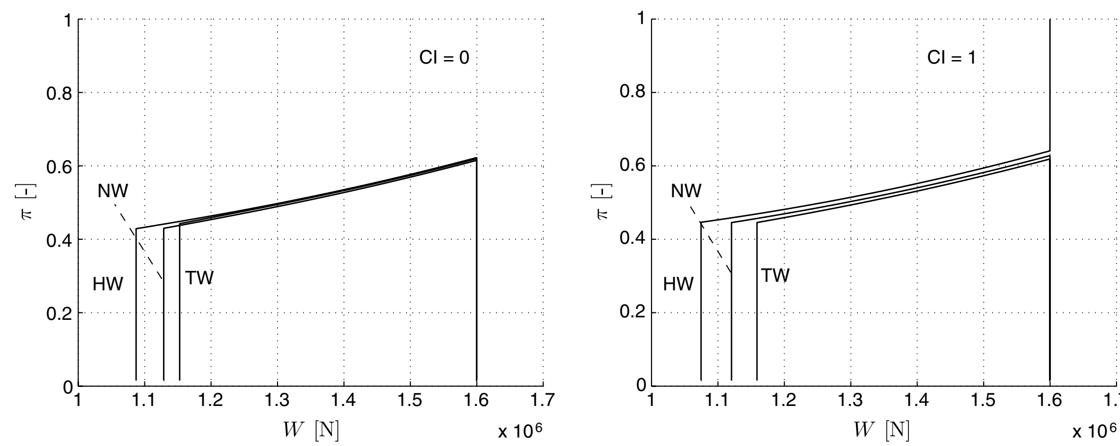


Fig. 2 Cost vs fixed flight time; TC (solid line) and DOC (dashed line) for $K = 0.5$ kg/s and TW $w = 20$ m/s: a) nominal case for $\mathrm{CI} = 1$ kg/s and $h = 10{,}000$ m, and b) critical case for $\mathrm{CI} = 0$ and $h = 9000$ m.



a) b)

Fig. 3 Optimal trajectories (Mach vs weight) for different wind conditions ($K = 0.5$ kg/s, $h = 10,000$ m): a) CI = 0 kg/s and b) CI = 1 kg/s.



a) b)

Fig. 4 Optimal control vs weight for different wind conditions ($K = 0.5$ kg/s, $h = 10,000$ m): a) CI = 0 kg/s and b) CI = 1 kg/s.

expected is the result that the optimal values of Mach numbers are larger for larger values of the CI (that is, larger values of the cost of time), corresponding to the smaller values of flight time.

The optimal control corresponding to the optimal trajectories represented in Fig. 3 is depicted in Fig. 4, being of the bang-singular-bang type (note the maximum-thrust arc in the case $\mathrm{CI} = 1$ kg/s and HW). The singular control decreases along the cruise, and its variation with the wind condition is very weak (in fact, it can be shown that π varies very slightly with speed and strongly with aircraft weight and with altitude).

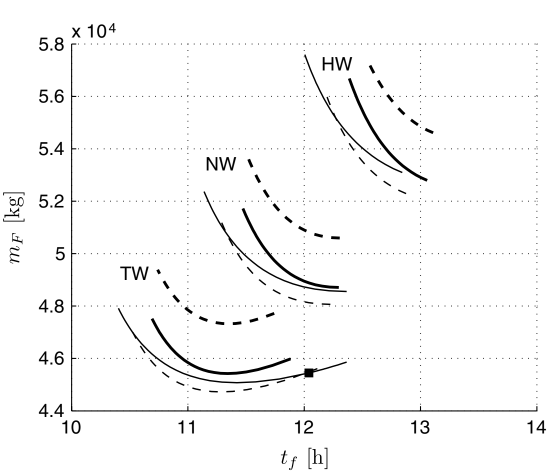


Fig. 5 Optimal fuel consumption vs optimal flight time for $K = 0.5$ kg/s, different wind conditions, and different altitudes: $h = 9000$ m (thin solid), 10,000 m (thin dashed), 11,000 m (thick solid), and 12,000 m (thick dashed).

2. *Optimal Flight Time and Fuel Consumption*

The tradeoff between fuel consumption and flight time is shown in Fig. 5 for different altitudes ($h = 9000$, 10,000, 11,000, and 12,000 m) and the given wind conditions. In each curve, the CI ranges from 0 to 3 kg/s, going from right to left (from large to small values of t_f). The flight time decreases as the CI increases (that is, as the cost of time increases). The flight time is considerably larger in the case of a HW as compared with the case of a TW; for example, for $\mathrm{CI} = 1$ kg/s and $h = 10,000$ m, the difference is approximately 1.5 h ($t_f = 10.97$ h for TW and $t_f = 12.49$ h for HW). The case of NW is the nominal case considered to define the scheduled time of arrival, so that t_f for NW is in fact t_s; for example, for $\mathrm{CI} = 1$ kg/s and $h = 10,000$ m, one has $t_s = 11.65$ h.

As the CI increases, one has that m_F increases for HW and NW, whereas it has a minimum value for TW (in fact, the three curves at each altitude have a minimum at $\mathrm{CI} + \Phi_{t_f} = 0$; that is, at $\mathrm{CI} = -0.5$, 0, and 0.5 kg/s). The fuel consumption is considerably larger for the case of a HW as compared with the case of a TW; for example, for $\mathrm{CI} = 1$ kg/s and $h = 10,000$ m, the difference is around 8600 kg ($m_F = 53,583$ kg for HW and $m_F = 44,978$ kg for TW).

The dot in the curve for TW and $h = 9000$ m (thin solid line) in Fig. 5 indicates that, to its right, one has the critical behavior (that part of the curve ranges from $\mathrm{CI} = 0$ to $\mathrm{CI} = 0.17$ kg/s). To show this last

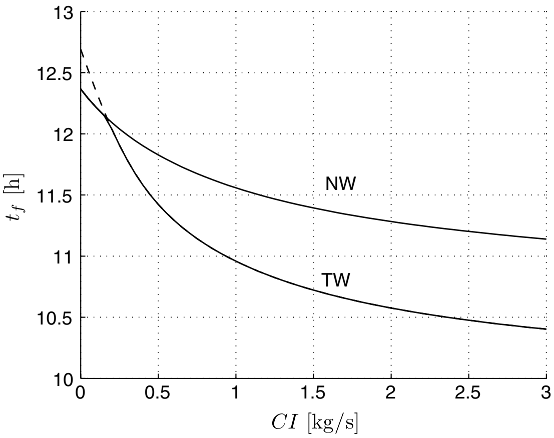


Fig. 6 Optimal flight time vs CI for TW and NW ($K = 0.5$ kg/s, $h = 9000$ m).

result more clearly, t_f vs CI is presented for $h = 9000$ m in Fig. 6, both for TW and for the nominal case of NW. For small values of CI, one has $t_f > t_s$, contrary to what one expected (the dashed part of the curve for TW would correspond to a negative arrival-error cost). For that range of CI, one has the kind of critical case analyzed in Sec. V, in which the optimal solution corresponds to a cruise with $t_f = t_s$ (final time equal to the nominal scheduled time of arrival); hence, the solution for TW coincides with the solution for NW.

3. *Minimum Total Cost*

Now, the minimum cost is quantified, obtained either by minimizing the TC or by minimizing the fuel consumption for $t_f = t_s$ (critical case, for which the analysis is made in Sec. VI.B). The minimum TC is represented in Fig. 7 as a function of CI for $h = 10,000$ m and for the given wind conditions: the TC increases quite linearly as the CI increases.

In Fig. 8, the effect of K is shown for $\mathrm{CI} = 0.5$ kg/s and $h = 10,000$ m. The part of the curve for TW to the right of the dot corresponds to the critical behavior, in which the cost is independent of K. Both for HW and for TW, the minimum TC increases as K increases (as expected).

The effect of the wind condition is seen in Figs. 7 and 8: the cost for HW is always considerably larger than for TW, since t_f and m_F are larger (see Fig. 5). For example, for $\mathrm{CI} = 1$ kg/s and $h = 10,000$ m, the TC increases about 17%, from 85,677 kg for TW to 100,063 kg for HW (see Fig. 7). Also, the difference between HW and TW increases as the CI increases.

It can be also shown that there is a best altitude at which the minimum cost is lowest; this best altitude is quite close to 10,000 m for $K = 0.5$ kg/s and for the three wind conditions considered: 9764 m for TW, 9915 m for NW, and 9928 m for HW.

4. *Critical Minimum-Cost Problem*

In this subsection, a parametric analysis is presented that quantifies the ranges of the parameters at which the critical case analyzed in Sec. V occurs. In Fig. 9, the effect of the arrival-error-cost parameter is shown for two altitudes ($h = 9000$ and 10,000 m), two values of the CI (0 and 0.5 kg/s), and TW ($w = 20$ m/s). The value of K at which the critical behavior shows up increases as CI and altitude increase. In this graph, one has that the optimum flight time is the one that minimizes the TC up to a value of K and, from there on, is equal to t_s (independent of K). This behavior appears for a certain value of K, and it remains as K increases.

The effect of the windspeed is shown in Fig. 10 for two altitudes ($h = 9000$ and 10,000 m), two values of the CI (0 and 0.5 kg/s), and $K = 0.5$ kg/s. One has that, for given values of h, CI, and K, there is a range of relatively small windspeeds in which one has the critical behavior; this range decreases as CI and altitude increase.

B. Minimum-Fuel Cruise with Fixed Arrival Time

In this section, the problem of minimum fuel consumption with fixed arrival time is analyzed, which, as already seen, corresponds to the critical case of the minimum-TC problem, although it is an interesting problem by itself, with practical operational application being an example of a four-dimensional (4-D) problem.

The minimum-fuel problem with fixed arrival time is studied as a minimum-DOC problem: the objective is to find the CI such that the flight time coincides with the fixed arrival time (see [3,5,7]). Since the arrival time is fixed, one does not have any arrival-error cost, thus, in this case, one sets $K = 0$. The same numerical procedure (described in Sec. IV) is also used now.

The singular arc that corresponds to this problem in the case of NW has been studied in [14]. In this section, the general trajectory (bang-singular-bang) is considered and wind effects are included. The optimal trajectories and control present characteristics analogous to those obtained before in the analysis of the minimum-cost problem.

The minimum fuel consumption as a function of the final time is shown in Fig. 11 for $h = 10,000$ m and for the given wind conditions. All curves in Fig. 11 present a minimum that corresponds to $\mathrm{CI} = 0$; these minima are the solutions of the minimum-fuel problem with free final time.

It can be also shown that there is a best altitude that provides lowest minimum fuel consumption; this best altitude is around 10,000 m: 9866 m for TW, 9960 m for NW, and 10,028 m for HW.

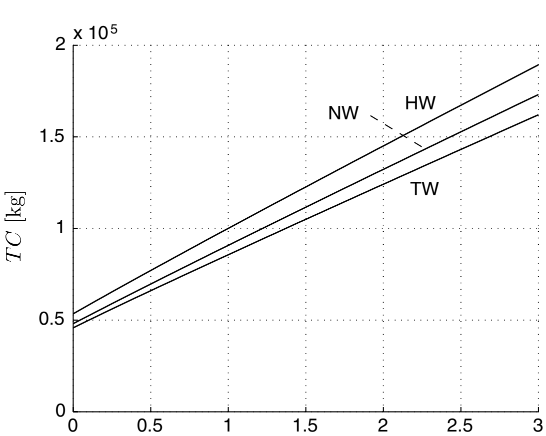


Fig. 7 Minimum TC vs CI for different wind conditions ($K = 0.5$ kg/s, $h = 10,000$ m).

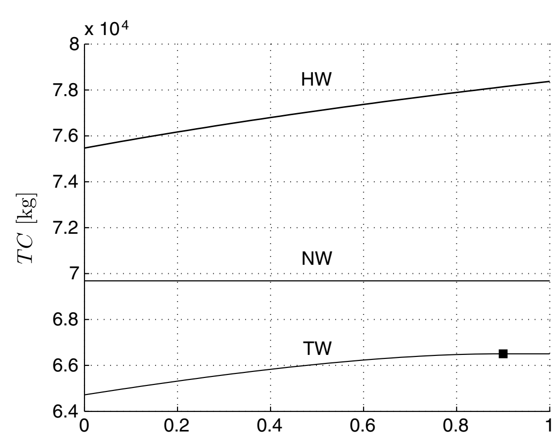
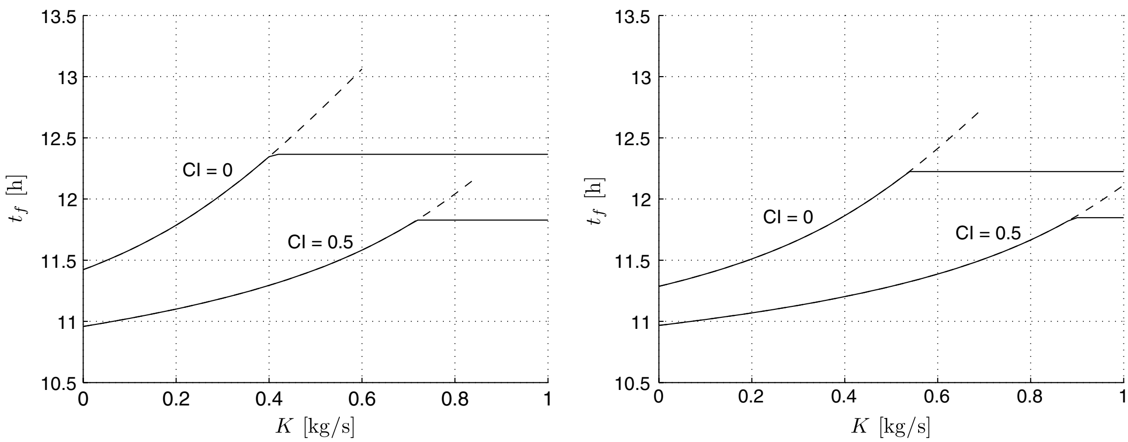
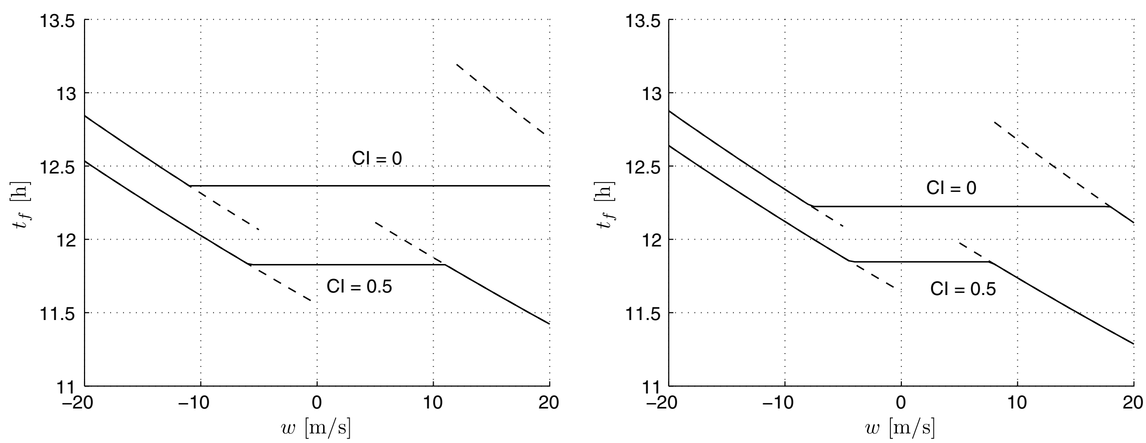


Fig. 8 Minimum TC vs arrival-error-cost parameter for CI = 0.5 kg/s, $h = 10,000$ m, and different wind conditions.



a) b)

Fig. 9 Optimal flight time vs arrival-error-cost parameter for different values of CI (0 and 0.5 kg/s), TW ($w = 20$ m/s), and different altitudes: a) $h = 9000$ m and b) $h = 10,000$ m.



a) b)

Fig. 10 Optimal flight time vs windspeed for different values of CI (0 and 0.5 kg/s), $K = 0.5$ kg/s, and different altitudes: a) $h = 9000$ m and b) $h = 10,000$ m.

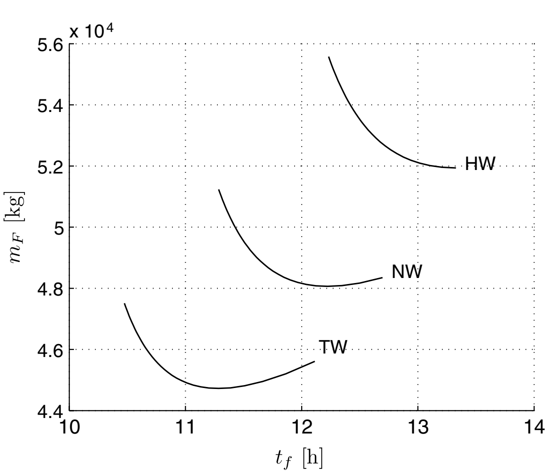


Fig. 11 Minimum fuel consumption vs flight time (minimum-fuel 4-D problem) for different wind conditions ($h = 10,000$ m).

VII. Conclusions

An analysis of minimum-cost cruise at constant altitude has been made, considering the DOC and the arrival-error cost associated to unexpected winds. The arrival-error cost has been defined so as to penalize both late and early arrivals in order to achieve high arrival-time accuracy.

The analysis has been made using the theory of singular optimal control. The case of arbitrarily given initial and final values of the speed has been considered. The optimal control is of the bang-singular-bang type, and the optimal trajectories for the values of speed considered are started by either a minimum or a maximum-thrust arc, followed by a singular arc, and finished by a minimum-thrust arc. The optimal speed laws (Mach number as a function of aircraft weight) that define those trajectories have been obtained, as well as the corresponding controls.

It has been shown that, for some values of the parameters of the problem, minimum cost is obtained when cruising so that the flight time coincides with the nominal scheduled time of arrival (that is, cruise with fixed final time); hence, optimum performance is obtained by making the arrival-error cost zero. One has this critical case at relatively small values of windspeed, small values of cruise altitude, small values of CI, and large values of the arrival-error-cost parameter. This problem, which is in fact a minimum-fuel problem with fixed final time, has been analyzed as a minimum-DOC problem with free final time.

An interesting result, from the operational point of view, is that there are best altitudes at which both the minimum TC and the minimum fuel consumption with fixed arrival time are smallest. Selecting these altitudes would improve cruise performance.

For concreteness, to define the arrival-error cost, a scheduled time of arrival corresponding to a nominal case with NW has been considered; however, one could have considered a nominal case with a given average wind as well. The arrival-error cost has been formulated as a piecewise linear function of the difference between the actual and scheduled flight times; the analysis of other types of arrival-error costs (as a function of the flight time) is left for future work.

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